Contemporary Mathematics
Volume 44, 1985

LECTURES ON 3-FOLD SIMPLE COVERINGS AND 3-MANIFOLDS

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To Professor Arthur Stone

In these lectures we will essentially study 3-fold simple branched coverings of the 3-sphere S^3 as a procedure for representing closed, orientable 3-manifolds ("3-manifolds" along the paper). After motivating our theme by a look to the case of surfaces, we will concentrate on 3-manifolds.

1. Introduction: the case of surfaces.

1.1 A simplicial map $f : M^n \to N^n$ between two compact, triangulated n-manifolds M and N is a *branched covering* if it is an ordinary covering out of the $(n-2)$-skeleton of N. The points of N whose preimage has less points than the number m of sheets of f, form a subcomplex R^{n-2} of N^n which is called the *branching set.* We will say that "f is an m-fold covering of N branched over R."

Given R^{n-2}, sub-complex of N^n, each covering of $N - R$ has a unique completion (cfr. [6]) called the *associated branched covering.* If B is the preimage of R we have the following diagram

$$\begin{tikzcd} M - B \arrow[r, hook] \arrow[d, "p'"'] & M \arrow[d, "p"] \\ N - R \arrow[r, hook] & N \end{tikzcd}$$

where p' is the original covering and p is the associated branched covering. Let $b \in B$ and let U be a neighborhood of $p(b)$ in N. Let V be the component of $p^{-1}U$ containing b. Then $V - B \to U - R$ is a u-fold covering. The minimal u for every U is the *branching index in* b. If this index is finite for every $b \in B$, then M is triangulated and B is a $(n-2)$-subcomplex. If R is a locally flat submanifold, then M is a manifold and B is a submanifold.

*Supported by "Comisión Asesora de investigación científica y técnica"

Thus in the branched covering $p: M \to N$, M is determined by $M - B$, and this manifold is determined by the monodromy

$$\omega : \pi_1(N - R) \to S_m$$

of the (unbranched) covering p', where S_m is the symmetric group of m indices.

1.2 Let K be the group generated by reflections on the sides of the triangle of $\mathbb{R}^2$ with vertices in $(0,0),(1,1)$ and $(1,0)$. The translations of vectors (4,0) and (0,4) generate a normal subgroup G of K. Let H be the normal subgroup of K which extends G by the reflection u through $(0,0)$. We have the commutative diagram

$$\begin{tikzcd} \mathbf{R}^2 \arrow[r] \arrow[d] & \mathbf{R}^2/H \cong S^2 \\ \mathbf{R}^2/G = T^2 \arrow[ur, "p"'] & \end{tikzcd}$$

Thus p is a 2-fold covering of the 2-sphere S^2 branched over the 4 points A_i of Figure 1. Here T^2 is the 2-torus, and S^2 is the quotient by u, the 180° rotation around the axis passing through the points $\tilde{A}_i$.

1.3 The group K/G acts on T^2 and this action projects to the action of K/H on S^2. The group K/H, being the full group of symmetries of S^2 fixing the set $\{A_1, A_2, A_3, A_4\}$, coincides with $\mathbb{Z}_2 \times D_4$, where D_4 is the dihedral group of order 8. The element t of this group which is a 90° rotation of S^2 sending A_1 to A_4 followed by a reflection on the plane of the points A_i, is the projection of a 90° rotation of T^2 around $(1,1)$ followed by reflection on $\tilde{H}'$. We call this homeomorphism $\tilde{t}$ and we see that the homeomorphism $\tilde{t}_* : H_1(T^2;\mathbb{Z}) \circlearrowleft$ is given by the equations:

$$[\tilde{t}_*\tilde{Q}, \tilde{t}_*\tilde{H}] = [\tilde{Q}, \tilde{H}]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1.4 The homeomorphism of $\mathbb{R}^2$ given by $(x,y) \mapsto (x, y-x)$ is isotopic to the one of Figure 2, which is a suitable translation on the shadowed vertical bands. This homeomorphism projects to the homeomorphisms $\tilde{v}^{-1} : T^2 \to T^2$ and $v^{-1} : S^2 \to S^2$ shown in Figure 3. Thus $\tilde{v}^{-1}$ is what is called a *negative Dehn-twist along* $\tilde{Q}'$ inducing the automorphism

$$[\tilde{v}_*^{-1}\tilde{Q}, \tilde{v}_*^{-1}\tilde{H}] = [\tilde{Q}, \tilde{H}] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

in homology. The homeomorphism v^{-1} is a *positive half-twist along* Q'.

1.5 The homeomorphism $v^{a_1} t v^{a_2} t \ldots v^{a_r} t$ of S^2, where a_i are integers, lifts to a homeomorphism of T^2 with matrix

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_r & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix},$$

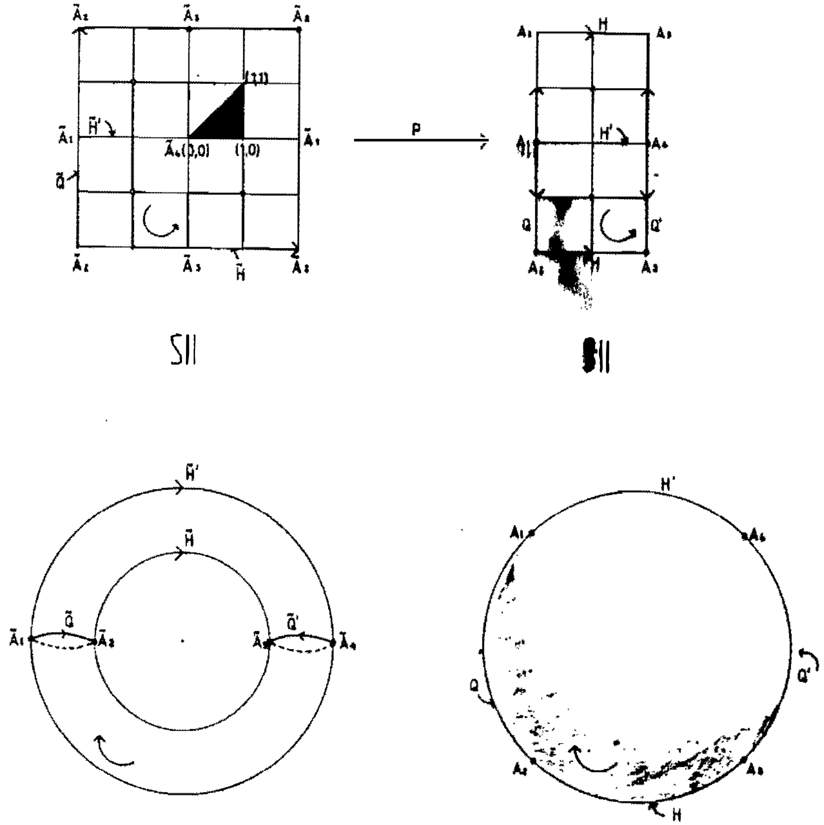


FIGURE 1

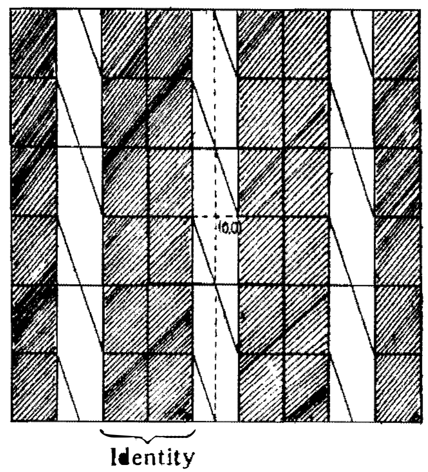


FIGURE 2

and is easy to check that

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_r}}.$$

The image of $\tilde{Q}$ by this homeomorphism $\tilde{f}(\alpha/\beta)$ is $\alpha\tilde{Q} + \beta\tilde{H}$. The projection of $\tilde{f}(\alpha/\beta)$ is called $f(\alpha/\beta)$. What we essentially have is that, up to isotopy, every homeomorphism of T^2 commutes with the involution u (see [4], [28]).

1.6 It is easy to see that every closed, orientable surface F_g of genus $g > 1$ is a 2-fold covering of S^3 branched over $2g+2$ points. But except for $g = 1, 2$, it is not true that every homeomorphism of F_g commutes with the covering involution (cfr. [2], [42], [7]).

1.7 We want to give another example of branched covering between surfaces, namely an irregular 3-fold covering, coming from the algebraic geometry of curves in $\mathbb{C}P^2$. Take a non singular cubic C in $\mathbb{C}P^2$ and a point p in its complement so that among the lines passing through p, exactly 6 are tangent to C (cfr. [3], p. 268). The set of lines through p is $\mathbb{C}P^1 \cong S^2$ and we have a map $g : C \to \mathbb{C}P^1$ which is 3 to 1 in general. But if $x \in \mathbb{C}P^1$ is tangent to C, $g^{-1}(x)$ consists of 2 points. The map g is an irregular covering, C is a torus and the preimage of a point in the branching set consists of a point of branching index 2 (the point of tangency) and a point of branching index 1.

2. Two-fold branched coverings of S^3.

2.1 The typical example is the 2-fold covering $P : S^3 \to S^3$ where the covering involution U is a 180° rotation around an axis of $S^3 = \mathbb{R}^3 + \infty$. Thus the branching set is the trivial knot T of S^3. More generally, given a link L of S^3 we consider the representation $\omega : \pi_1(S^3 - L) \to S_2$ which sends the meridians to

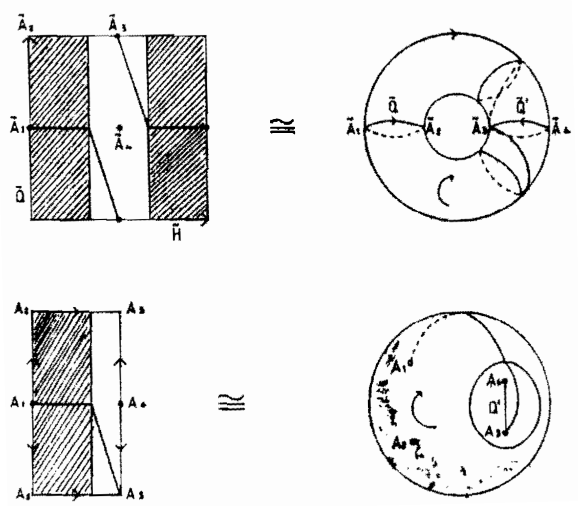


FIGURE 3

the transposition $(12) \in S_2$. The manifold $M(L, \omega)$ is the *2-fold covering of S^3 branched over L.*

2.2 We consider the torus T^2 of Figure 1 embedded in $S^3 = \mathbb{R}^3 + \infty$ so that the axis of the involution U passes through the points $\tilde{A}_i$, thus inducing the involution u in T^2 as in Figure 4.

The core of the solid torus V bounded by T^2 is a *strongly-invertible knot* K(or *link*, if we use a number of tori T^2, which are left invariant by U). The image of T^2 under $P : S^3 \to S^3$ is also shown in Figure 4. We select the curve $\tilde{H}$ on T^2 so that it is homologous to zero in $S^3 - \mathrm{Int}\,V$. The manifold $M(K, \alpha/\beta)$ obtained by Dehn-surgery (with new meridian $\alpha\tilde{Q} + \beta\tilde{H}$) on K is $V \bigcup_{\tilde{f}(\alpha/\beta)} (S^3 - \mathrm{Int}\,V)$ and this is a 2-fold covering of $S^3 \cong B^3 \bigcup_{f(\alpha/\beta)} (S^3 - \mathrm{Int}\,B^3)$, the branching set is obtained replacing the arcs A_1A_2, A_3A_4 by the *rational tangle*  defined as in Figure 5 (for $\alpha/\beta = 38/7 = 5 + \frac{1}{2+\frac{1}{3}}$).

This shows that *every 3-manifold obtained by Dehn-surgery on a strongly invertible link is a 2-fold branched covering of S^3 [31], [43].*

In particular, the lens space $L(\alpha, \beta)$ is the result of Dehn-surgery α/β on the trivial knot in S^3 which is strongly invertible. Thus $L(\alpha, \beta)$ is a 2-fold covering

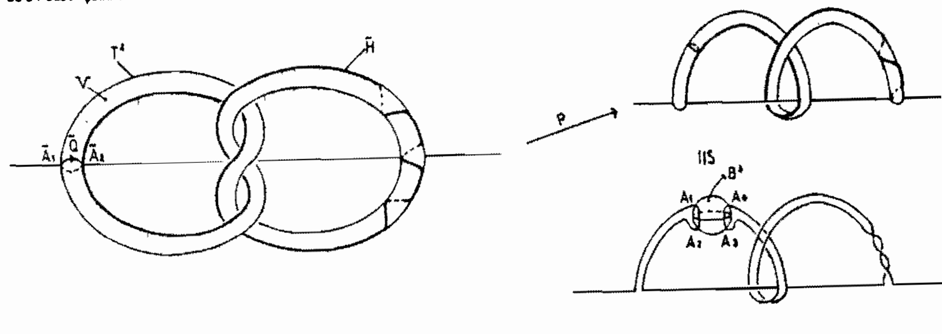


FIGURE 4

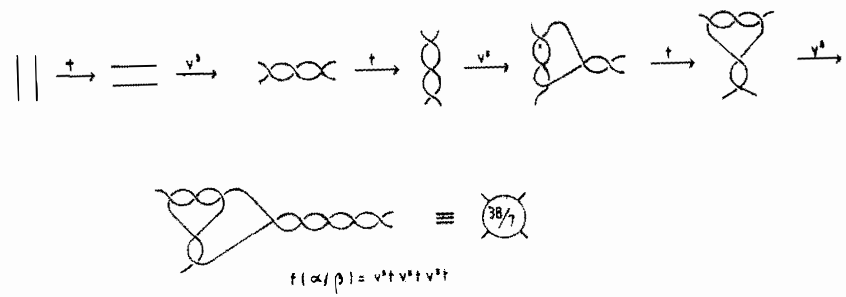
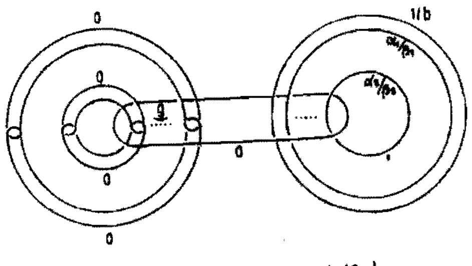


FIGURE 5

of S^3 branched over the *rational link*  $= R(\alpha/\beta)$. This is a knot when α is odd [38].

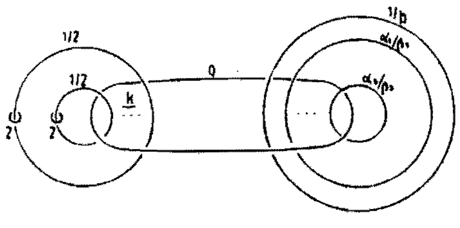
2.3 It is easy to see (cfr. [35], p.114) that the orientable Seifert manifolds have the Dehn-surgery description shown in Figures 6 and 7. The links of Figure 6 and 7 are symmetric with respect to an axis. Then from 2.2 it follows that these manifolds are branched coverings as depicted in Figure 8. Thus the manifolds with base S^2 or a non orientable surface are 2-fold branched coverings of S^3. The ones with orientable base of genus g are 2-fold branched coverings of $g\#S^1\times S^2$. Similar results hold for the graph-manifolds of Waldhausen (cfr. [28], [32]).

2.4 We remark that there are 3-manifolds obtained by Dehn-surgery on a non-invertible link which are 2-fold branched coverings of S^3 (cfr. [48], [9]). On



$(0og/b;\ \alpha_1/\beta_1, \ldots, \alpha_s/\beta_s)$

FIGURE 6



$(Onk/b;\ \alpha_1/\beta_1, \ldots, \alpha_s/\beta_s)$

FIGURE 7

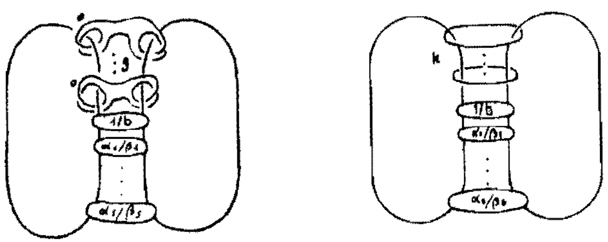


FIGURE 8

the other hand, there are 3-manifolds which are not 2-fold branch coverings of S^3, for instance $S^1\times S^1\times S^1$ [7] (cfr. [27], [28], [19], [24], [9], [37]).

2.5 Though S^3 is a 2-fold branched covering of S^3 in essentially one way [45], and the same happens with some other manifolds ($n\#S^1\times S^2$ [45]; lens spaces [20]), there are manifolds with different presentations as 2-fold branch coverings spaces (see [42], [25], [49], [44], [32], [8], [50], [36]); for instance, the torus knot $\{3,7\}$ and the pretzel knot $\{-2,3,7\}$ have the same associated 2-fold covering (cfr. also [32], [21; problem 3.41], [22]).

2.6 Since there are 3-manifolds without periodic homeomorphisms [41], it is impossible to represent the class of all 3-manifolds as regular coverings of S^3. This is why we look to irregular coverings.

3. Irregular 3-fold coverings.

3.1 The irregular 3-fold coverings were considered long ago by Heegaard [10] (cfr. [52],[40]) in relation with algebraic surfaces. If we project $\mathbb{C}^3$ onto $\mathbb{C}^2$ by $(x,y,z)\mapsto(y,z)$, and we restrict this map to the algebraic affine surface F given by the equation $x^3+yx-z=0$, we obtain a map $p: F\to\mathbb{C}^2$ with the following properties: the preimage of each point $(y,z)\in\mathbb{C}^2$ consists of the points $(x_i,y,z), 1\le i\le 3$, where x_i is a solution of $x^3+yx-z=0$ (see Figure 9; cfr. the beautiful book [3, p.233]).

This map is a 3-fold branched covering, the ramification being the discriminant $D\equiv 27z^2+4y^3=0$. The preimage of the points of D consists of 2 points except for $(0,0)$ which has only one. Thus this covering is irregular. The restriction of p to $p^{-1}S^3$, where $S^3=\{(y,z)\in\mathbb{C}^2|y\bar{y}+z\bar{z}=1\}$, defines a 3-fold covering $g: S^3\to S^3$ branched over the trefoil 3_1 (Figure 10), because $p^{-1}S^3$, being the boundary of a regular neighborhood of $p^{-1}(0,0)$ in F, must be S^3. The monodromy $\omega: \pi_1(S^3-3_1)\to S_3$ sends meridians to transpositions, because the branching indices of the two points which form the preimage of $P\in 3_1$ must be 2 and 1. Since ω must be onto, we have, up to conjugation in $S_3, \omega(x)=(12)$, $\omega(y)=(13)$ [and $\omega(z)=(23)$]. The covering $p^{-1}: D^4\to D^4$ where $D^4=\{(y,z)\in\mathbb{C}^2|y\bar{y}+z\bar{z}\le 1\}$ is equivalent to the cone of $g; S^3\to S^3$.

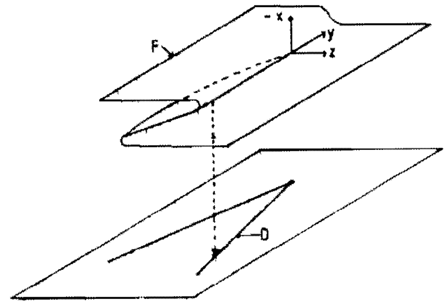


FIGURE 9

3.2 We are going to study 3-fold coverings $p: M^3 \to S^3$ branched over a link $L \subset S^3$, given by a transitive representation

$$\omega : \pi_1(S^3 - L) \to S_3$$

which send meridians to transpositions. We call this sort of coverings *3-fold simple coverings.* The example of 3.1 is typical. The monodromy group of the covering is the dihedral group of 6 elements S_3 and the group of covering transformations is trivial. A nice way for defining ω, given L, is due to Ralph Fox [5]. Three colors (say G=green, R=red, B=blue) are used to color the bridges of the link. This is done in such a way that three colors that meet at an overcrossing are either all the same, or all distinct. Then a representation is defined on the meridians, $G \to (12)$, $R \to (13)$, $B \to (23)$. The color condition guarantees that the defining relations are sent to the identity in the Wirtinger presentation. If at least two colors are used, the representation is transitive. A representation defined in this way is called a *colored knot or link.* Since $p: M \to S^3$ is not the quotient by a group acting on M, it is a little difficult to visualize p. We describe now a 3-fold simple covering $p: S^3 \to S^3$ as the starting step for proving that every 3-manifold is a 3-fold simple covering.

3.3 We describe $p: S^3 \to S^3$ as the double $2p'$ of a map $p': D^3 \to D^3$. The map p' is depicted in Figure 11, where the ball D^3 is the result of performing identifications in the sides of the cylinder as depicted. The map p' is the result of folding D^3 along the axis B_{12} and B'_{13}, exactly as one does when folding a letter.

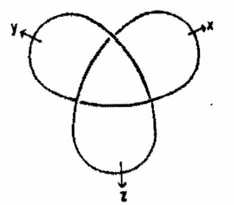


FIGURE 10

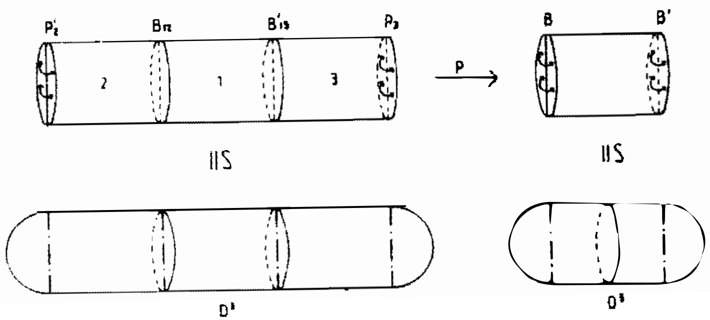


FIGURE 11

Then $p'^{-1}B$ is the union of B_{12} (branching index 2) and P_3 (branching index 1). We will call $B_{12} \cup B'_{13}$ the *branching cover* and $P'_2 \cup P_3$ the *pseudobranching cover.* Note the important fact that p' near B_{12} (and B'_{13}) works like a 2-fold branched covering. We use this and the facts about 2-fold covering, displayed in section 2, to represent any 3-manifold as a 3-fold simple covering.

Now the map $p = 2p : 2D^3 \to 2D^3$ is very easy to understand. The branching set is composed of two unlinked trivial knots and its preimage consists of four unlinked trivial knots.

The coverings q (of 3.1) and p are clearly different in as much as they have different branching sets. But they are closely related as we shall see later. The multiplicity of representations of a manifold as a simple 3-fold covering of S^3 is typical as we shall see.

3.4 Consider some 3-manifold M obtained by surgery on a link L of S^3. To fix the ideas, assume L is the non invertible knot 8_{17} of Figure 12, and M is obtained by surgery on it. Take 8_{17} and place it between the axis P'_2 and B_{12} of Figure 11. We then reflect 8_{17} through B_{12} to get a 180° rotated copy $8'_{17}$ of 8_{17} between B_{12} and B'_{13}. Finally we make the connected sum of 8_{17} and $8'_{17}$ to get the strongly invertible knot $8_{17}\#8'_{17}$ with respect to the axis B_{12}, as depicted in Figure 13.

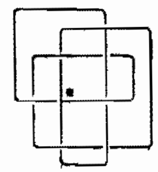


FIGURE 12

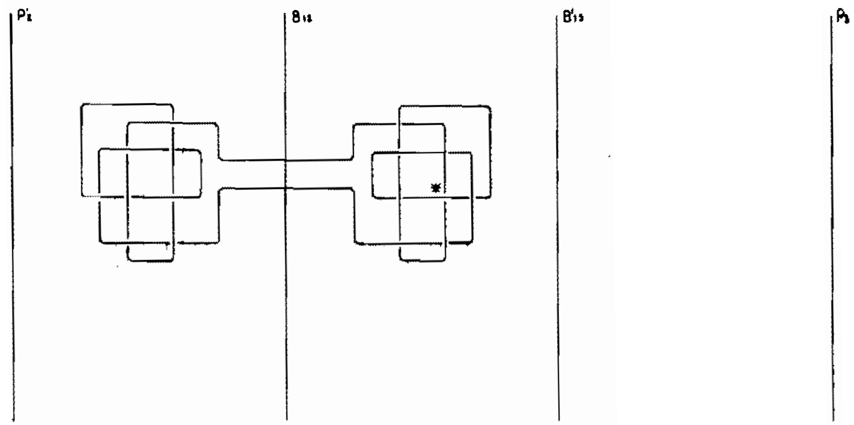


FIGURE 13

We now look for crossings of L such that local changes of underpasses by overpasses replace L by a trivial split link (in $L = 8_{17}$, the crossing marked with a star). We now introduce around those crossings a set T of unlinked trivial knots (Figure 14) which we make symmetric with respect to B'_{13}. Now the manifold M can be obtained by surgery on $8_{17}\#8'_{17} \cup T$ because doing the Hempel's trick [11] on T we perform a local change of an underpass by an overpass and thus $8_{17}\#8'_{17}$ becomes the original knot 8_{17}. Thus we can assume that M is obtained by α/β-surgery on $8_{17}\#8'_{17}$ and ± 1 surgery on T.

Projecting Figure 14 through p we get Figure 15 and the image of $8_{17}\#8'_{17} \cup T$ is the union of the arcs $p(8_{17}\#8'_{17}), p(T)$ with their endpoints on the branching set $B \cup B'$. Replacing the rational tangles $N(p(8_{17}\#8'_{17})) \cap B$, $N(p(T)) \cap B'$ [where $N(\cdots)$ stands for regular neighborhoods] by the tangles α/β and ± 1, the branching set B change to $\overline{B}$ and the corresponding 3-fold covering changes by α/β-surgery on $8_{17}\#8'_{17}$, (± 1)-surgery on T, and two surgeries on the regular neighborhood of the two arcs which are contained in the preimage of $p^{-1}(p(8_{17}\#8'_{17} \cup T))$. Thus the covering space changes from S^3 to M and this shows that M is a simple 3-fold covering of S^3 branched over $\overline{B}$.

Since every 3-manifold can be obtained by surgery on a link of S^3 [23] [46], the above argument shows that *every 3-manifold is a 3-fold simple cover of* S^3 [12] [29] (cfr. [13], [30], [18], [33], [1]) *and thus it corresponds to a colored link.*

3.5 As an example, consider the 3-torus T^3 which is the result of 0-surgery on the borromean rings. Figure 16 depicts the borromean rings. Figure 17 shows the branching set.

3.6 The last example is interesting because the branching set is the boundary of two orientable surfaces. This suggests a method to obtaining many examples of colored links (and of branched coverings). Thus let G and R be a pair of bounded disjoint surfaces in S^3, which we can think of as discs with bands in "general position with respect to a plane" so that all the singularities of the

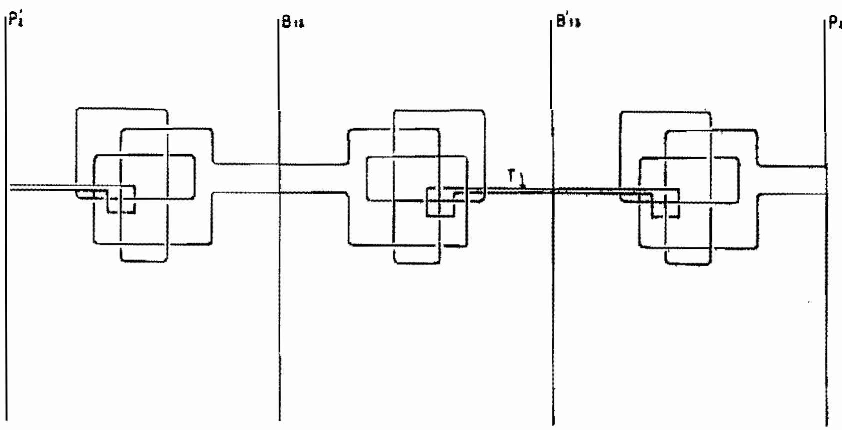


FIGURE 14

projection are the type shown in Figure 18. If the boundary of G^2 is colored green except for those parts that are under R^2, and the boundary of R^2 is colored red except for those parts under G^2, we obtain a colored link which we call *natural.* Then the proof of 3.4 shows that *every 3-manifold is a 3-fold covering of* S^3 *branched over the natural colored link associated to connected surfaces* G, R (cfr. [14]), moreover, the surfaces *can be supposed to be orientable* [17] (cfr. [16], [21]). The proof consists in showing that we can obtain an analogous situation to the one of Figure 14 but with *even-integer* framings, but we will not give the details here.

4. Moves.

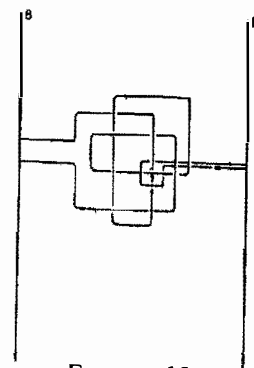


FIGURE 15

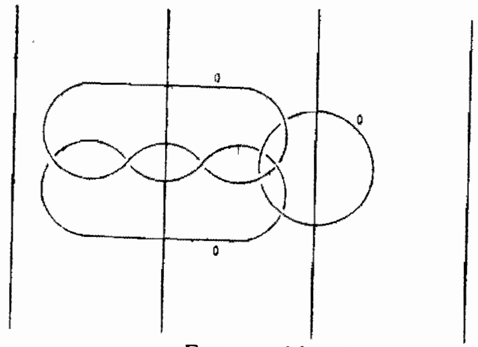


FIGURE 16

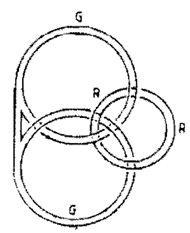


FIGURE 17

4.1 Every link L in S^3 has a *n-bridge presentation*, i.e. a triad (S^3, L, S^2), where the 2-sphere S^2, which separates S^3 into two balls D_1^3, D_2^3, is such that $D_i \cap L$ is a collection of n unknotted and unlinked arcs properly embedded in D_i, for $i = 1, 2$. If $p: M^3 \to S^3$ is a covering branched over L, then the sphere S^2 lifts to a closed orientable surface which defines a Heegard splitting of M^3.

4.2 If $p: M^3 \to S^3$ is a simple 3-fold covering, the genus g of $F_g = p^{-1}S^2$ is $n-2$. Hence if L is a 2-bridge link, the manifold M^3, having genus zero, must be S^3. This is what happens with the example of 3.1 because the trefoil knot has two bridges.

4.3 The class of 2-bridge links coincides with the class of rational links $R(\alpha/\beta)$ (see 2.2) and $R(\alpha/\beta)$ can be colored exactly when 3 divides α. Hence if in the projection of a colored knot we perform the move of Figure 19 the covering manifold does not change because we are cutting out a ball upstairs and regluing it back again in a different way. This follows from 4.2 [7], [25], [26], [18].

4.4 Using these moves one can represent a 3-manifold M as a simple 3-fold covering of S^3 in many different ways (for instance, if $M = S^3$ in at least as many as rational links $R(3\alpha/\beta)$). The moves $M(3\alpha/\beta)$ are a consequence of the moves $M(\pm 3)$ (see Figure 20), as it is very easy to check. We have long ago posed the following

Problem. *Find a set of moves which do not change the covering and such that if two colored links have the same cover, they are related by a finite sequence of moves.*

One would like that $M(\pm 3)$ be the required set of moves [25] [7], but this is not the case [51]. The interest of the problem is that if that set does exist, we can translate the classification problem of 3-manifolds in the problem of classification of colored links under some combinatorial moves.

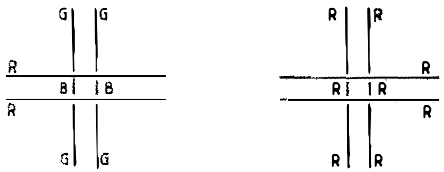


FIGURE 18

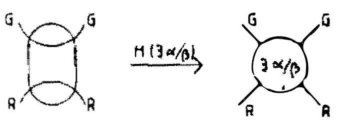


FIGURE 19

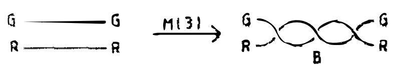


FIGURE 20

4.5 We invite the reader to apply moves $M(\pm 3)$ in the places of Figure 21 and to realize how the knots split in the components shown under them. We may say that a colored link L is *separable* if, using moves $M(\pm 3)$, it is possible to obtain a link $L_1 \cup L_2$, where L_1 is colored in green and L_2 in red (cfr. [25], [7], [34]).

It is easy to see that every colored closed braid with 2 or 3 strings is separable [25]. However, there are non-separable links, because *if the colored link L is separable into $L_1 \cup L_2$, the corresponding covering manifold M is a 2-fold covering of S^3 (branched over any of the connected sums $L_1 \# L_2$*[25], [7]). The proof of this statement is easy, and its converse is not true in general [51]. Thus, the colored link of Figure 17 [7] (see [27], [28]) is non-separable, while the manifolds which are associated to the knots of Figure 21 are 2-fold coverings of S^3; namely $S^3, \mathbb{R}P^3$, $L(3,1), L(4,1)$, $L(5,2), S^1 \times S^2$, and the homology 3-sphere discovered by Poincaré (cfr. [47]).

Problem. *If the covering manifold associated to the colored link L is S^3, is L separable?*

Fox Conjecture. [7] *A colored link, having a simply connected associated covering manifold, is separable.*

This conjecture implies the Poincaré conjecture, because by the strong Smith conjecture, the associated covering manifold, being a 2-fold branched covering, must be S^3. On the other hand, the Poincaré conjecture implies the Fox conjecture exactly when the answer to the last problem is "yes".

4.6 It is easy to see that every colored link can be converted in a knot by a number of applications of moves $M(\pm 3)$. Hence we have that *every 3-manifold is a 3-fold simple covering of S^3 branched over a knot* [12], [29]. The simpler knot for S^3 is the trefoil. The branching set can also be converted in a pure closed braid [13] or even in a fibred link (using moves $M(\pm 3)$ to get the conditions of Stallings [39] for fibred closed braids). This fibration can be lifted to an open book decomposition of the associated covering 3-manifold.

5. Some applications.

5.1 We want to give a simpler and constructive proof, due to Hilden and the author, of the result [14] that *every 3-manifold is a simple 3-fold covering of S^3 branched over a knot so that the branching cover bounds an embedded disc* (the pseudo branching cover, though, can be knotted). The starting point is a 3-manifold M represented as a simple 3-fold covering of S^3 branched over the natural colored link associated to the connected surfaces G and B. The manifold M is constructed taking three copies $(S^3_K, G_K, B_K), K = 1,2,3$ of the triple (S^3, G, B), splitting along $G_K \cup B_K$ and gluing $G_1^{\pm}$ with $G_2^{\mp}$ and $B_2^{\pm}$ with $B_3^{\mp}$. Thus the branching cover bounds $G_2^+ \cup B_2^-$. We want to convert $G_2^+ \cup B_2^-$ in two discs which we will connect later. This is obtained using the moves which depend on $M(\pm 3)$ (see Figures 22 and 23):

To understand this, take a ball containing the clasp of Fig. 23 (see Figure 24) in the copy (S_2^3, G_2', B_2'). Cutting open this ball along $B_2' \cup G_2' \cup C_2'$, we obtain the ball depicted in Figure 25, where $a_2, b_2, c_2, b_2', c_2'$, are are the preimages of $\bar{a}_2$

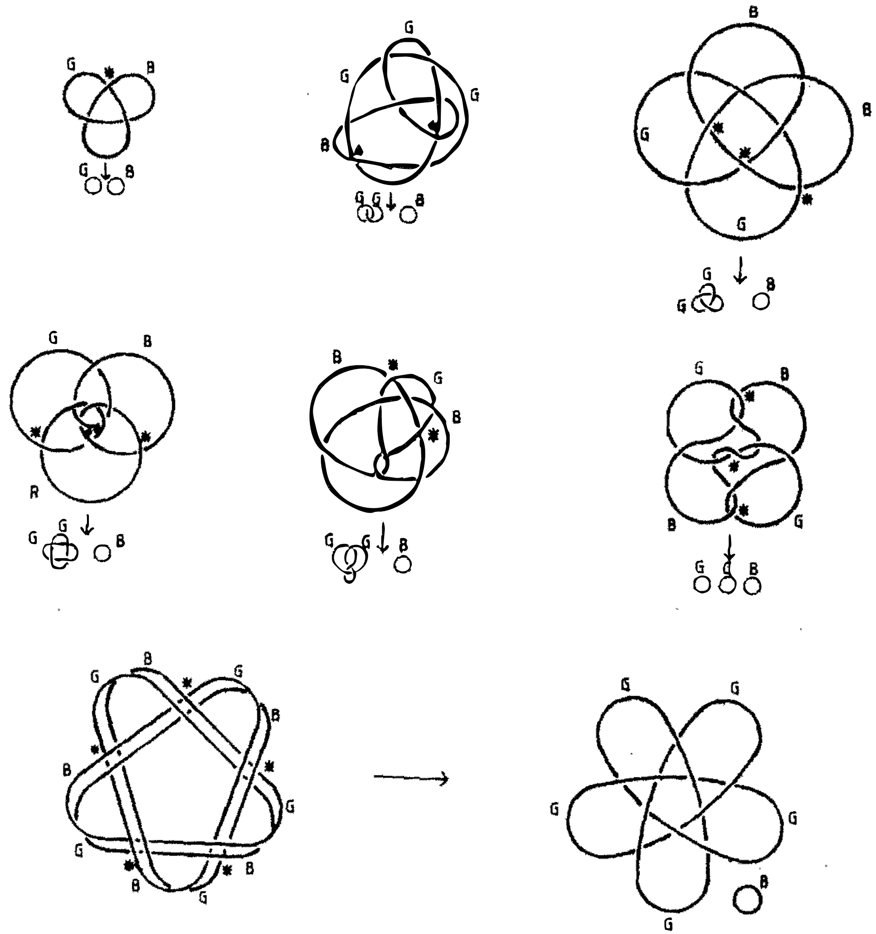


FIGURE 21

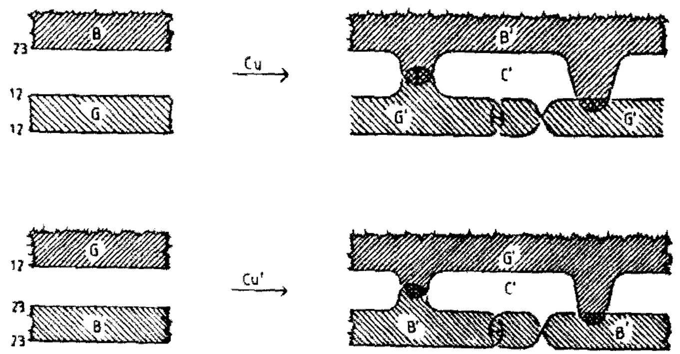


FIGURE 22

(for details of this process see [25], [26]). The arcs marked b (resp. p) belong to the branching (pseudo branching) cover.

Note that $a_3 = c_3$ because $G_3'^-$ and $G_3'^+$ must be identified. Also $a_3 = b_2$, $c_3 = b_2'$ because $B_3'^+$ is identified with $B_2'^-$. Hence $b_2 = b_2'$. By analogous reason $c_2 = c_2'$. Thus $(B_2'^-/b_1 = b_2') \cap (G_2'^+/c_2 = c_2') = \emptyset$ and $(B_2'^-/b_2 = b_2') \cup (G_2'^+/c_2 = c_2')$ bound the branching cover. In the preimage of the ball containing the clasp, we have the situation depicted in Figure 26. Thus the clasp downstairs is an unclasp in the branching cover.

Take now the ball containing the crossing of Figure 23 in the copy (S_2^3, G_2', B_2'), depicted in Figure 27. Cutting open this ball along $B_2' \cup G_2' \cup C_2'$ we obtain the ball depicted in Fig. 28, where $B_2'^-$ and $G_2'^+$ touch along an arc and thus they are connected.

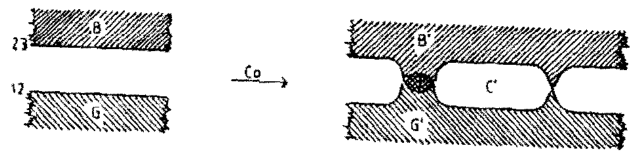


FIGURE 23

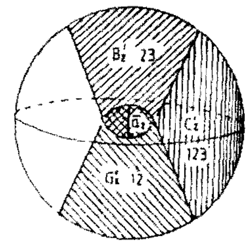


FIGURE 24

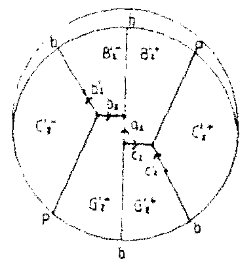


FIGURE 25

Using move C_u, it is possible to cut one by one all the bands of $G_2'^{+}$. The similar move C_u' would cut all the bands of $B_2'^{-}$. Finally move C_o would connect $B_2'^{-}$ with $G_2'^{+}$. This proves our theorem.

A nice consequence of the last theorem is that ***every 3-manifold is parallelizable*** because so it is out of the disc D bounded by the branching cover since here the projection is a local homeomorphism. On the other hand, since $\pi_2(SO(3)) = 0$ there is no obstruction to extend the parallelization to a ball containing the disc D [14].

5.2 We now show that *any 3-manifold is the pullback of any 3-fold simple covering $p : S^3 \to S^3$ and some smooth map $\Omega : S^3 \to S^3$ transversal to the*

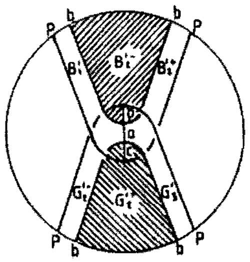


FIGURE 26

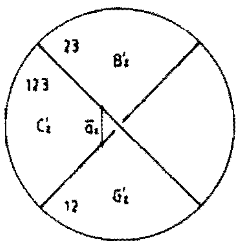


FIGURE 27

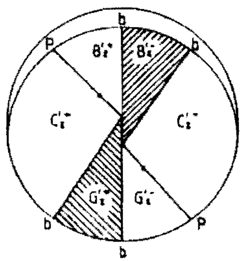


FIGURE 28

branching set of p. [17]. The starting point is a 3-manifold M represented as a simple 3-fold covering of S^3 branched over the natural colored link L associated with the *orientable* surfaces G and R. If we send green (red) meridians to the letter $g(r)$, we have a homeomorphism $\omega_* : \pi_1(S^3 - L) \to F_2$, where F_2 is the free group in the letters g, r. Note that to prove that ω_* is a homomorophism, we need both G and R to be orientable. The composition $\lambda\omega_*$, where $\lambda(g) = (12)$, $\lambda(r) = 13$ defines the representation giving rise to the covering $g : M \to S^3$ branched over L.

Let $S^1 \vee S^1$ be two circles identified along a point p. Since $S^1 \vee S^1$ is aspherical, there is a continuous map $\omega : S^3 - L \to S^1 \vee S^1$, realizing ω_*. We extend ω to $\Omega' : S^3 \to D^2 \vee D^2$ in the most natural way, where $D^2 \vee D^2$ are two discs identified along p.

We now take $p : S^3 \to S^3$ a 3-fold simple covering branched along the colored link K, and we embedded $D^2 \vee D^2$ in such a way that it cuts K transversaly in the centers of the two discs (Figure 29). The composition of Ω' and this embedding can be approximated by a smooth map. $\Omega : S^3 \to S^3$ transversal to K. Then clearly $\Omega^{-1}(K) = L$. Thus the pullback of p and Ω is a 3-fold simple covering of S^3 branched over L, corresponding to the representation

$$\pi_1(S^3 - L) \xrightarrow{\omega_*} \pi_1(S^1 \vee S^1) = F_2 \xrightarrow{i_*} \pi_1(S^3 - K) \xrightarrow{\eta} S_3$$

where i is the natural embedding and η is the representation corresponding to p. But $\eta i_* = \lambda$ hence that pullback is $g : M \to S^3$. This proves the theorem. An interesting question is to find the relationship between two $\Omega, \tilde{\Omega}$ giving rise to the same 3-manifolds.

5.3 A consequence of the last Theorem is the following partial sharpening of a beautiful theorem of Hilden [15] (cfr. [17]). *It is possible to embed any 3-manifold M in $S^3 \times D^2$ so that the composition with the projection in the first factor is a 3-fold simple covering.* Thus M is a sort of Riemann space as in the initial example of 3.1 with which we started our study.

In fact, during the proof of the last theorem we found a map $\Omega' : S^3 \to D^2 \vee D^2$. We have a 3-fold covering $p : E^2 \to D^2 \vee D^2$ branched over the centers of the discs (see Figure 30, compare with Figure 11).

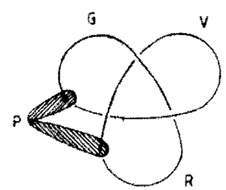


FIGURE 29

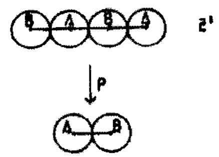


FIGURE 30

Clearly the pullback of p and Ω' is the covering $g : M \to S^3$. We then have the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\tilde{\Omega}^{1'}"] \arrow[d, "g"'] & E^2 \arrow[d, "p"] \\ S^3 \arrow[r, "\Omega^{1'}"] & D^2 \vee D^2 \end{tikzcd}$$

The required embedding $M \to (S^3 \times D^2)$ is given by $(g, i\tilde{\Omega}')$, where $i : E^2 \to D^2$ is an embedding. Note that the number of points in $g^{-1}x$ coincides with the number of points of $\tilde{\Omega}' g^{-1}x$. This proves the theorem.

Note that $S^3 \times D^2$, being part of the boundary of D^6, embeds in S^5, hence we obtain the classical result of Morris Hirsch that every 3-manifold embeds in S^5.

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